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Technical Notes

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Evidence-Based Fuzzy Approach for the Safety Analysis of Uncertain Systems

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I. Introduction

THE application of Dempster–Shafer theory for combining multiple sources of evidence to handle the uncertainties present in engineering systems is well established. In this work, an evidence-based fuzzy approach is presented for the safety analysis of uncertain engineering systems in the presence of multiple sources of evidence. The α-cut approach is used to represent the fuzzy membership functions of the uncertain parameters. The existence of large epistemic uncertainty information for each of the uncertain parameters is assumed to be available in the form of interval-valued data from multiple sources. The fuzzy membership function of the response of the system, such as the margin of safety, is computed by applying fuzzy arithmetic to the mathematical formulation of the system. A new procedure is introduced to calculate bounds on the response of the system, such as the margin of safety. A new methodology, termed the weighted fuzzy theory for intervals (WFTI), is proposed for combining evidence when different credibilities are associated with the various sources of evidence. The application of the proposed methods is illustrated by considering the design of a welded beam involving multiple uncertain parameters. The results obtained using the proposed WFTI method are verified to converge to those obtained using the fuzzy approach when all the credibilities tend to have an identical value of unity.

II. α-cut Representation

When engineering systems are too complex with ill-defined or imprecise information present in the geometry, material properties, external effects, or boundary conditions, the fuzzy approach can be used to find the response of the system. A fuzzy quantity or a fuzzy number X is described either by its membership function $X = \{(x, \mu_X(x)), x \in I_X\}$ or by the union of its α cuts $X = \cup_\alpha \cdot \alpha \cdot [x_l^{(\alpha)}, x_u^{(\alpha)}]$, where $\alpha \in [0, 1]$. To extend the mathematical laws of crisp numbers to fuzzy theory, we can use the extension principle, which provides a methodology that fuzzifies the parameters or arguments of a function, resulting in computable fuzzy sets. Thus, for convenience of numerical computations, fuzzy numbers are expressed as sets of lower and upper bounds of a finite number of α-cut subsets. Corresponding to a level of α (α cut), the value of x is extracted in the form of an ordered pair with a closed interval $[x_l, x_u]$. The α cut can be taken anywhere ranging from $\alpha = 0$ (total uncertainty) to $\alpha = 1$ (total certainty). An interval number is represented as an ordered pair $[x_l, x_u]$, where $x_l \leq x_u$. When $x_l = x_u$, the interval is called a fuzzy-point interval (e.g., $[a, a]$). Thus, membership functions are constructed in terms of intervals of confidence at several levels of α cuts.

The fuzzy numbers thus defined are known as intervals. Once the intervals or ranges of a fuzzy quantity corresponding to specific α cuts are known, the system response at any specific α cut can be found using interval analysis. Thus, in the application of a fuzzy approach to uncertain engineering problems, interval analysis can be used. With fuzzy quantities expressed in interval form, fuzzy arithmetic operations can be carried out using interval operations at each of the n α levels independently.

III. Fuzzy Approach for Combining Evidence

When evidence from multiple sources are available regarding the uncertainty of a system, Dempster–Shafer theory has traditionally been used to combine the evidence [1,2]. Dempster–Shafer theory is considered as a generalization of probability theory in which probabilities are assigned to sets instead of mutually exclusive events. In Dempster–Shafer theory, evidence can be associated with multiple sets of events. By combining evidence from multiple sources, Dempster–Shafer theory provides the lower and upper bounds, in the form of belief and plausibility, for the probability of occurrence of an event. In this work, an uncertain parameter is modeled as a fuzzy variable, and the available evidence on the ranges of the uncertain parameter, in the form of basic probability assignments (BPAs), is represented in the form of membership functions of the fuzzy variable. The membership functions constructed from the available evidence from multiple sources are added as multiple fuzzy data to find the combined membership of the uncertain or fuzzy parameter. The resulting combined membership function of the fuzzy parameter is then used to estimate the lower and upper bounds of any response quantity of the system, (such as the margin of safety or margin of failure) in the context of the safety analysis of an engineering system.

IV. Computation of Bounds on the Margin of Failure/Safety

A procedure is outlined for computing the bounds on the margin of failure of the system based on the membership function of the margin of failure curve shown in Fig. 1a. Let A represent the area under the membership function curve until the margin of failure equals zero, and let B indicate the area under the membership function value curve for the margin of failure greater than zero. The lower and upper bounds for realizing the margin of failure (MF) greater than zero can be expressed as

$$\text{lower bound} = 0, \qquad \text{upper bound} = \frac{B}{A + B} \qquad A \gg B \quad (1)$$

Similarly, the lower and upper bounds on the margin of safety (MS) of the system can be expressed as (see Fig. 1b)

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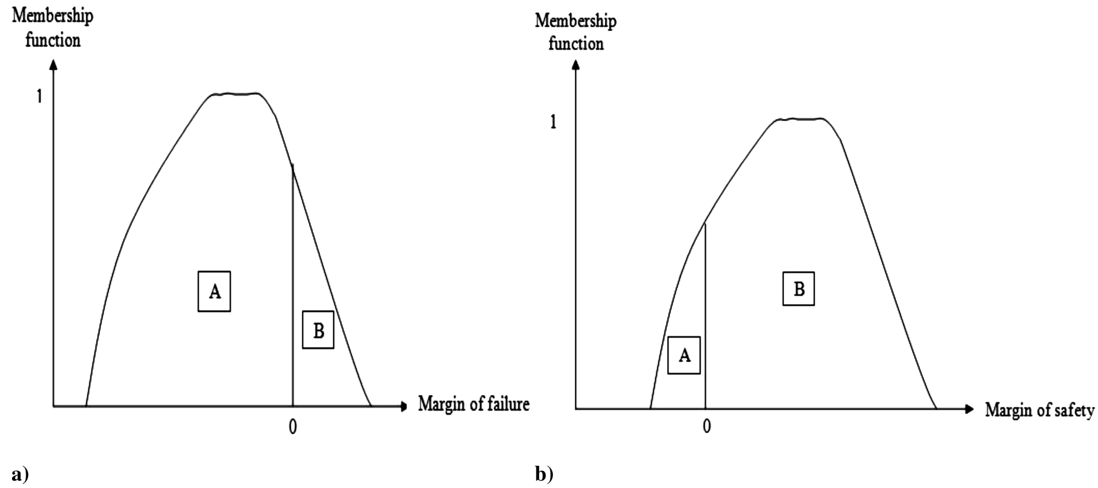


Fig. 1 Membership functions of the margins of failure and safety.

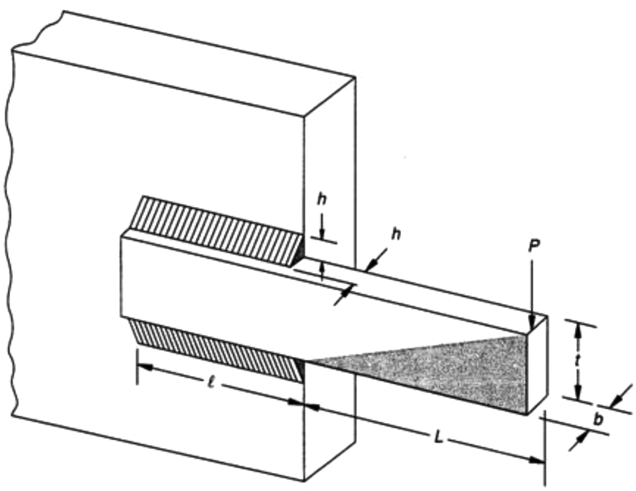


Fig. 2 A welded beam.

$$\text{lower bound} = \frac{B}{A+B}, \qquad \text{upper bound} = 1 \qquad A \ll B \tag{2}$$

V. Illustrative Example

To illustrate the methodology of combining evidence using a fuzzy approach, the safety/failure analysis of a welded beam is considered [3]. The beam is of length L with cross-sectional dimensions t and b and is welded to a fixed support, as shown in Fig. 2. The weld length is l on both the top and bottom surfaces and the beam is required to support a load P. The weld is in the form of a triangle of depth h. The maximum shear stress developed in the weld, τ, is given by

$$\tau = \sqrt{(\tau')^2 + 2\tau'\tau''\cos\theta + (\tau'')^2} \tag{3}$$

where

$$\tau' = \frac{P}{\sqrt{2}.h.l} \tag{4}$$

$$\tau'' = \frac{MR}{J} = \left[P\left(L + \frac{l}{2}\right)\right] \times \left\{\sqrt{\left[\frac{l^2}{4} + \left(\frac{h+t}{2}\right)^2\right]} \Bigg/ 2\left(\frac{hl}{\sqrt{2}}\left[\frac{l^2}{12} + \left(\frac{h+t}{2}\right)^2\right]\right)\right\} \tag{5}$$

$$\cos\theta = l \Bigg/ 2\sqrt{\left[\frac{l^2}{4} + \left(\frac{h+t}{2}\right)^2\right]} \tag{6}$$

The beam is considered unsafe if the maximum shear stress in the weld is greater than the permissible stress of $\tau_{\max}$ for the data [1]: $P = 6000$ lb, $L = 14$ in., $E = 30 \times 10^6$ psi, $\tau_{\max} = 13,600$ psi, $h = 0.3437$ in., $l = 8.149$ in., and $t = 8.273$ in. The bounds on the margin of safety and margin of failure of the welded beam are computed for two types of data. In the first type, the uncertain parameters are assumed to be fuzzy with triangular membership functions. In the second type, the ranges of the uncertain parameters are assumed to be available in the form of evidence from multiple sources.

A. With Triangular Membership Functions

A MATLAB program is developed to implement the fuzzy arithmetic required to handle the uncertain parameters and obtain an assessment of the likelihood of the maximum induced shear stress exceeding the specified permissible value. The length of the weld (l) and the height of the weld (h) are treated as the uncertain parameters, with x_1 and x_2 denoting the multiplication factors that define the uncertainties of these parameters. The membership functions of x_1 and x_2 are assumed to be triangular [4,5]. The permissible stress $\tau_{\text{allowable}}$ is assumed to be a fuzzy quantity with a triangular membership function in the range of $\pm 15\%$ of $\tau_{\max}$ ($0.85\tau_{\max}$ to $1.15\tau_{\max}$ psi) with $\tau_{\max} = 13,600$ psi. The margin of safety is calculated as the fuzzy difference between $\tau_{\text{allowable}}$ and $\tau_{\max}$, as shown by the solid line in Fig. 3. Similarly, the margin of failure is calculated as the fuzzy difference between $\tau_{\max}$ and $\tau_{\text{allowable}}$, as shown by the dotted line in Fig. 3. Equations (1) and (2) are used to find the lower and upper bounds on the margin of failure as well as the margin of safety based on the curves shown in Fig. 3, to obtain

$$0 < \text{MF} < 0.24873 \quad \text{and} \quad 0.75127 < \text{MS} < 1.0$$

B. With Membership Functions Based on Evidence from Multiple Sources

The safety/failure analysis of the welded beam is considered for two cases. In the first case, the membership functions are assumed to be triangular, whereas they are assumed to be trapezoidal in the second case. The range of triangular membership function for $\tau_{\text{allowable}}$ is assumed as $\pm 15\%$ of $\tau_{\max}$ ($0.85\tau_{\max}$ to $1.15\tau_{\max}$) with center at $\tau_{\max}$, and the range of trapezoidal membership function is assumed to be $\pm 15\%$ of $\tau_{\max}$ ($0.85\tau_{\max}$ to $1.15\tau_{\max}$) at $\alpha = 0$ and $\pm 3\%$ of $\tau_{\max}$ ($0.97\tau_{\max}$ to $1.03\tau_{\max}$) at $\alpha = 1$. The length of the weld

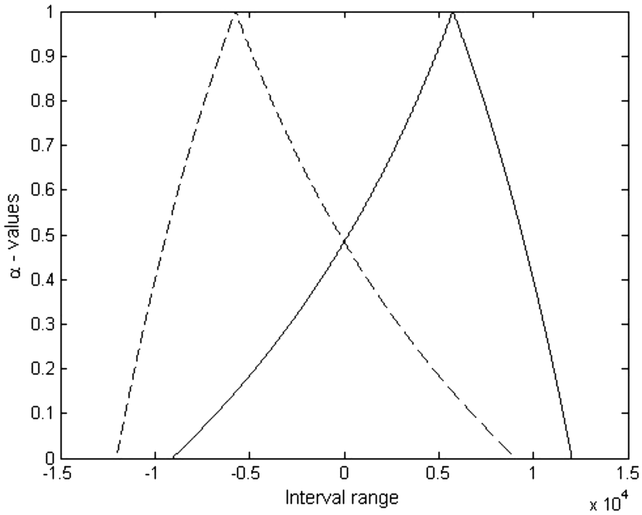


Fig. 3 Membership functions of the margins of safety and failure.

(l) and the height of the weld (h) are the uncertain parameters, with x_1 and x_2 denoting the multiplication factors that indicate their respective uncertainties. It is assumed that three sources of evidence provide possible ranges or intervals of x_1 and x_2 along with the corresponding BPAs (as in the case of Dempster–Shafer theory), as given in Table 1. The BPAs are normalized so that the maximum value corresponds to a membership value of 1 for modeling uncertain factors as fuzzy quantities, and the resulting fuzzy descriptions of factors x_1 and x_2 are assumed to be available from sources of evidence 1, 2, and 3. The three evidence sources (fuzzy descriptions) available for each of the factors x_1 and x_2 are added to obtain the combined fuzzy representations of x_1 and x_2 shown in Figs. 4a and 4b, respectively. These combined fuzzy representations are approximated. (smoothed by neglecting the valleys in the curves of Figs. 4a and 4b to avoid multiple disjointed ranges of the variables corresponding to any specific value of α cut in an α-cut representation of the fuzzy quantities). The fuzzy maximum induced shear stress in the weld, $\tau_{\max}$, is computed using the fuzzy parameters x_1 and x_2. A triangular membership function is assumed for the allowable shear stress. The fuzzy margin of safety and margin of failure are computed to obtain the curves shown in Fig. 5a. Similarly, in the second case, when a trapezoidal form of membership function is assumed for the allowable shear stress (τ_{allow}), the fuzzy margins of safety and failure are obtained as shown in Fig. 5b.

Table 1 Evidence for the uncertain factors x_1 and x_2 from sources S_1, S_2, and S_3

x_1	Source 1 (S_1)	Interval	[0.7,0.8]	[0.8,1.1]	[1.0,1.2]	[1.2,1.3]
		BPA	0.1	0.4	0.4	0.1
	Source 2 (S_2)	Interval	[0.7,0.9]	[0.8,1.0]	[1.0,1.2]	[1.1,1.3]
		BPA	0.1	0.4	0.3	0.2
	Source 3 (S_3)	Interval	[0.9,1.1]	[1.0,1.2]	[1.2,1.3]	——
		BPA	0.3	0.4	0.3	——
x_2	Source 1 (S_1)	Interval	[0.8,0.9]	[0.9,1.1]	[1.0,1.2]	[1.2,1.3]
		BPA	0.1	0.4	0.4	0.1
	Source 2 (S_2)	Interval	[0.7,0.9]	[0.9,1.0]	[1.0,1.2]	[1.1,1.3]
		BPA	0.2	0.4	0.2	0.2
	Source 3 (S_3)	Interval	[0.9,1.1]	[1.0,1.2]	[1.2,1.3]	——
		BPA	0.4	0.4	0.2	——

C. Bounds on Margins of Safety and Failure

The lower and upper bounds on the margins of safety and failure are computed using the methodology outlined in Sec. IV. The results are shown in Table 2. The Dempster–Shafer theory is also used to combine the evidence in each case. In Dempster–Shafer theory, the criteria $\tau_{\text{allowable}} \leq \tau_{\max}$ and $\tau_{\text{allowable}} > \tau_{\max}$ are used for the margin of safety and margin of failure, respectively, to obtain the results indicated in the last row of Table 2.

VI. Weighted Fuzzy Theory for Interval-Valued Data from Multiple Sources with Different Credibilities

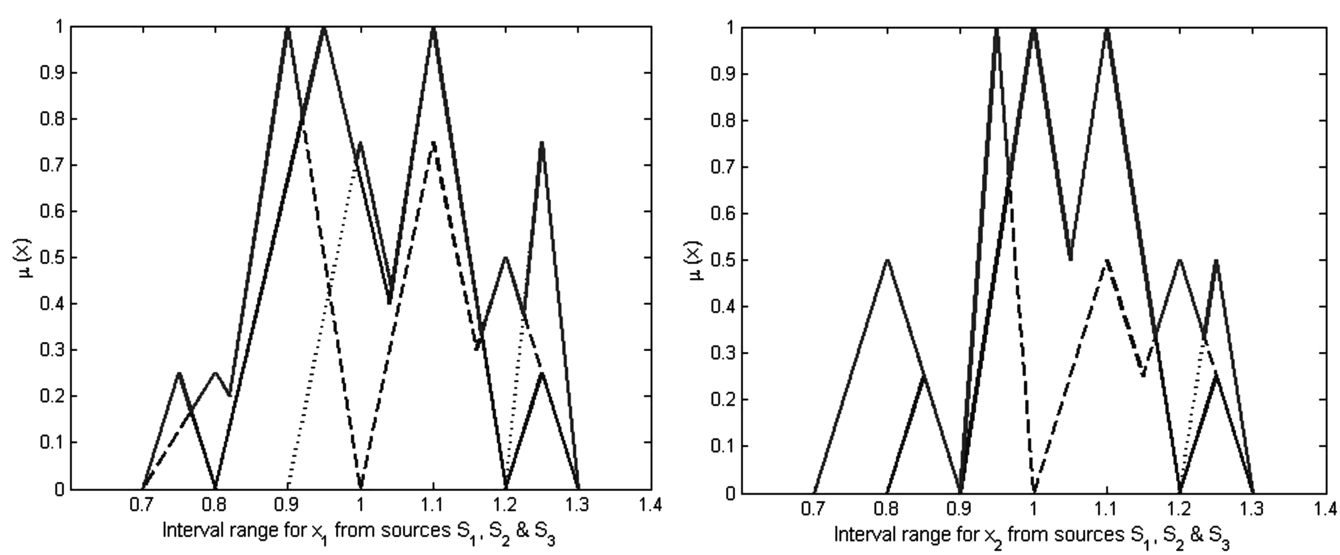
In practice, the credibilities of different sources of evidence for the uncertain parameters may be different. To consider the credibilities associated with the sources of evidence, a procedure termed the weighted fuzzy theory for intervals (WFTI) is proposed in this work for determining the lower and upper bounds on the margins of safety and failure. The general procedure is described by the following steps:

1) Normalize the evidence from each source before considering the credibilities of that source.

2) Multiply the normalized evidence by the credibilities of the corresponding sources. Each of the credibilities is assumed to lie in the range of 0 to 1.

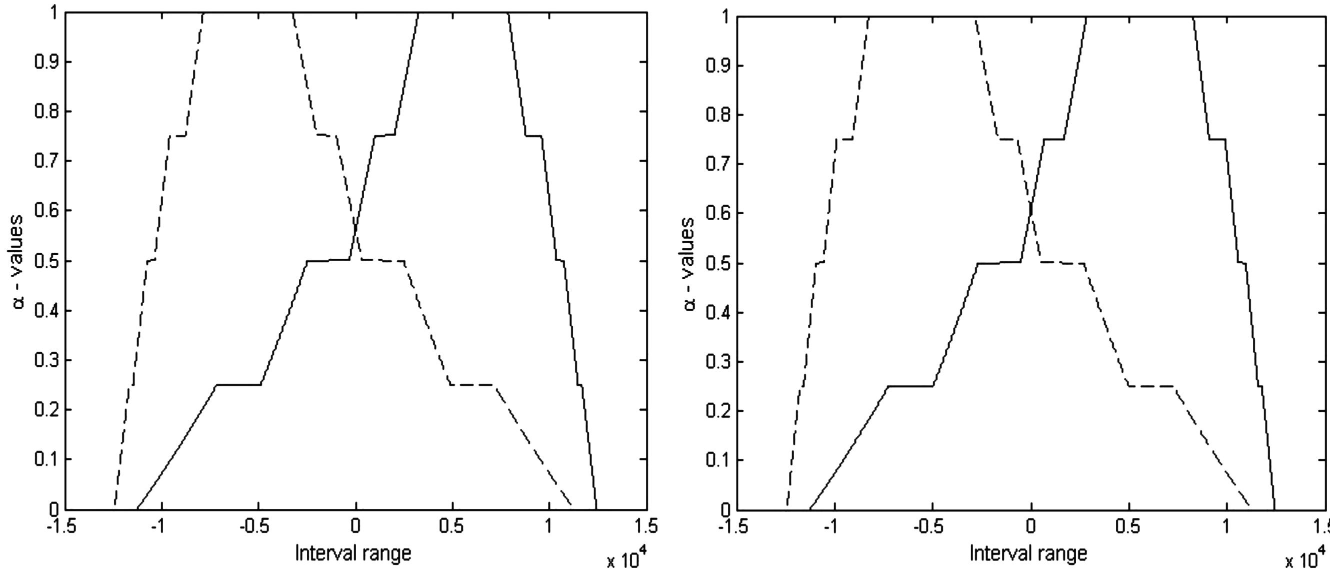
3) Obtain the combined outer envelope of the fuzzy membership function of each uncertain parameter by superposing all the membership functions (normalized evidence) of the uncertain parameter given by the various sources.

4) Compute the α cuts of the combined outer envelopes (or the combined fuzzy membership functions).



a) For x_1 **b) For x_2**

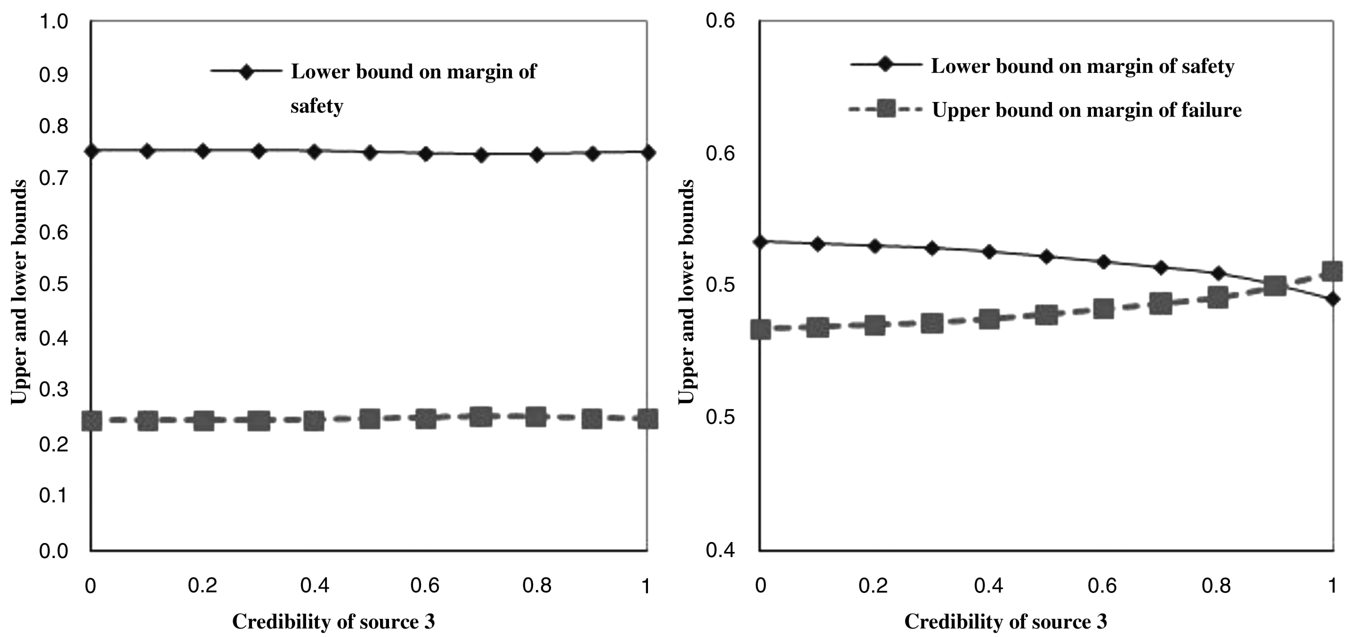
Fig. 4 Combined fuzzy membership functions of x_1 and x_2 from sources S_1, S_2, and S_3.



a) For x_1

b) For x_2

Fig. 5 Alpha-cut representation margins of safety and failure.



a)

b)

Fig. 6 Variations of lower bound on the margin of safety and upper bound on the margin of failure with the credibility of source S_3.

5) Calculate the response or output of the system using the algebra of interval numbers or parameters.

6) Determine the lower and upper bounds on the response or output parameter using the procedure described in Sec. IV.

A MATLAB program is developed to incorporate the WFTI procedure to include the credibility information of the various sources in the calculation of lower and upper bounds on the margins of safety and failure. The length of the weld (l) and the height of the weld (h) are considered to be the uncertain parameters, with x_1 and x_2 denoting the multiplication factors for these parameters. Three sources of evidence are assumed to provide possible ranges or intervals of x_1 and x_2 along with the corresponding BPAs, as given in

Table 2 Lower and upper bounds on the margins of safety and failure

Membership function of allowable shear stress $\tau_{\text{allowable}}$	Margin of safety		Margin of failure	
	Lower bound	Upper bound	Lower bound	Upper bound
Triangular	0.74966	1	0	0.25034
Trapezoidal	0.75215	1	0	0.24785
Dempster–Shafer theory	1	1	0	0

Table 3 Evidence for the uncertain factors x_3 and x_4 from sources S_1, S_2, and S_3

x_3	Source 1 (S_1)	Interval	[0.8,0.9]	[0.9,1.1]	[1.0,1.2]	——
		BPA	0.2	0.4	0.4	——
	Source 2 (S_2)	Interval	[0.7,0.9]	[0.9,1.0]	[1.0,1.2]	[1.1,1.2]
		BPA	0.1	0.5	0.2	0.2
	Source 3 (S_3)	Interval	[0.8,1.0]	[1.0,1.2]	[1.1,1.3]	——
		BPA	0.3	0.5	0.2	——
x_4	Source 1 (S_1)	Interval	[0.7,0.8]	[0.8,1.1]	[1.0,1.2]	[1.2,1.3]
		BPA	0.2	0.3	0.3	0.2
	Source 2 (S_2)	Interval	[0.7,0.9]	[0.8,1.0]	[1.0,1.2]	[1.1,1.3]
		BPA	0.1	0.5	0.2	0.2
	Source 3 (S_3)	Interval	[0.9,1.1]	[1.0,1.2]	[1.2,1.3]	——
		BPA	0.1	0.5	0.4	——

Table 1. The credibility of source S_3 is assumed to be less than 1 and several cases are studied by varying the credibility of source S_3 from 0 to 1 in increments of 0.1. Figure 6a shows the variations of the lower and upper bounds on the margin of safety and the margin of failure with varying values of the credibility of source S_3. To further demonstrate the WFTI approach, the welded-beam problem is reconsidered by assuming the length of the weld (l), height of the weld (h), depth of the cantilever (t), and length of the cantilever (L) to be uncertain, with x_1, x_2, x_3, and x_4 indicating their corresponding multiplication factors. The evidence for x_1 and x_2 are assumed to be the same as those given in Table 2. The evidence for x_3 and x_4 are assumed to be as indicated in Table 3.

Figure 6b shows the variations of the lower bound on the margin of safety and the upper bound on the margin of failure with varying values of the credibility of source S_3. It is found that the variation in the lower/upper bound on the margin of safety/failure as the credibility of source 3 increases depends on the evidence distribution on the interval ranges from source 3, the effect of uncertain parameters on the margin of safety/failure, and the number of uncertain parameters. The difference between the lower and upper bounds for both margin of safety and margin of failure increases with an increase in the number of uncertain parameters used in the analysis. The validity of this statement is intuitively obvious because the uncertainty of the system increases with an increase in the number of uncertain parameters used in the analysis.

VII. Conclusions

The sum of the lower bound on the margin of safety and the upper bound on the margin of failure is found to be equal to 1 in all cases, as expected. As the number of α cuts used in the numerical computation of the bounds on the margins of safety and failure change, the values of the computed bounds are found to vary, and for higher values, the bounds are found to converge to the values reported in this work. Irrespective of the number of sources of evidence, when the assumed fuzzy membership function of the allowable shear stress changes from the triangular to the trapezoidal shape, the lower and upper bounds tend to shrink the ranges of the margins of safety and failure. The widening or shrinking of the ranges of the margins of safety and failure is observed to depend on the available evidence and the influence of the uncertain parameters on the output or response parameter of the system. In general, the procedure proposed for considering the credibilities of the various sources of evidence (WFTI) is applicable for combining evidence to evaluate the safety/failure of any uncertain system in the presence of evidence on the uncertain parameters from different sources. The methodology presented in this work provides an alternative framework for combining evidence from multiple sources using fuzzy theory.

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